

On one integrable system with a cubic first integral

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Abstract

Recently one integrable model with a cubic first integral of motion has been studied by Valent using some special coordinate system. We describe the bi-Hamiltonian structures and variables of separation for this system.

1 Introduction.

The aim of this note is to consider one particular two-dimensional integrable model defined by natural Hamilton function

$$H = T + V = \sum_{i,j=1}^2 g_{ij}(q_2) p_i p_j + V(q_1, q_2) \quad (1.1)$$

with the metric depending on one variable, and cubic additional integral of motion with the leading terms

$$H_2 = p p_1^3 + 2q T p_1 + \dots, \quad p \in \mathbb{R}, \quad q \geq 0. \quad (1.2)$$

According to [7], relevant metrics are described by a finite number of parameters and lead to a large class of models mainly on the manifolds \mathbb{S}^2 and \mathbb{H}^2 . By suitable choices of the parameters entering the construction these systems are globally defined and contain as special cases the known systems of Goryachev-Chaplygin, of Goryachev, and of Dullin and Matveev, see [7].

In [6] we introduce a natural Poisson bivector depending on two arbitrary functions, which allows us to describe similar family of integrable system with cubic additional integral of motion in spherical coordinates. Below we rewrite this bivector in the coordinate system used in [7], calculate the corresponding variables of separation and prove that equations of motion are linearized on genus three non-hyperelliptic curve.

2 Settings

In this section we recall some necessary facts about natural bi-integrable systems on Riemannian manifolds admitting separation of variables in the Hamilton-Jacobi equation [2, 5, 6]

Let Q be a n -dimensional Riemannian manifold. Its cotangent bundle T^*Q is naturally endowed with canonical invertible Poisson bivector P , which has a standard form in fibered coordinates $z = (q_1, \dots, q_n, p_1, \dots, p_n)$ on T^*Q

$$P = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \{f, g\} = \langle P df, dg \rangle = \sum_{i=1}^{2n} P_{ij} \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial z_j}, \quad (2.1)$$

In order to calculate the variables of separation for the given integrable system with integrals of motion H_1, \dots, H_n in involution

$$\{H_i, H_j\} = 0, \quad i, j = 1, \dots, n,$$

in the bi-Hamiltonian set-up we have to solve equations

$$[P, P] = [P', P'] = 0, \quad (2.2)$$

where $[\cdot, \cdot]$ means a Schouten bracket, and

$$\{H_i, H_j\}' = 0, \quad i, j = 1, \dots, n, \quad \{f, g\}' = \langle P' df, dg \rangle, \quad (2.3)$$

with respect to the Poisson bivector P' . Then we have to calculate the so-called Nijenhuis operator (or hereditary, or recursion operator)

$$N = P' P^{-1}. \quad (2.4)$$

If N has, at every point, the maximal number of different functionally independent eigenvalues u_1, \dots, u_n , they may be treated either as action variables (integrals of motion) or as variables of separation for this dynamical system [2, 5].

Separation of variables for natural integrable systems with higher order integrals of motion always involves generic canonical transformation of the whole phase space. The definition of the natural Hamiltonians (1.1) is non-invariant with respect to such transformations of coordinates on the whole phase space (cotangent bundle).

In the situation, when habitual objects (geodesics, metrics and potentials) lose their geometric sense, and the remaining invariant equations (2.2, 2.3) have infinite number of solutions, the notion of the natural Poisson bivectors becomes de-facto a very useful practical tool for the calculation of variables of separation [5, 6]. It is an experimental fact supported by all the known constructions of the variables of separation on the sphere. We try to draw attention to this experimental fact in order to find a suitable geometric explanation of this phenomenon.

Similar to the natural Hamilton function on T^*Q , natural Poisson bivector P' is a sum of the geodesic Poisson bivector P'_T and the potential Poisson bivector defined by a torsionless (1,1) tensor field $\Lambda(q_1, \dots, q_n)$ on Q associated with potential V [5, 6]:

$$P' = P'_T + \begin{pmatrix} 0 & \Lambda_{ij} \\ -\Lambda_{ji} & \sum_{k=1}^n \left(\frac{\partial \Lambda_{ki}}{\partial q_j} - \frac{\partial \Lambda_{kj}}{\partial q_i} \right) p_k \end{pmatrix}. \quad (2.5)$$

The geodesic Poisson bivector P'_T is defined by $n \times n$ geodesic matrix Π on T^*Q :

$$P'_T = \begin{pmatrix} \sum_{k=1}^n x_{jk}(q) \frac{\partial \Pi_{jk}}{\partial p_i} - y_{ik}(q) \frac{\partial \Pi_{ik}}{\partial p_j} & \Pi_{ij} \\ -\Pi_{ji} & \sum_{k=1}^n \left(\frac{\partial \Pi_{ki}}{\partial q_j} - \frac{\partial \Pi_{kj}}{\partial q_i} \right) z_k(p) \end{pmatrix}. \quad (2.6)$$

In fact, for the given matrix Π functions x, y and z are completely determined by the equations

$$[P, P'_T] = [P'_T, P'_T] = 0, \quad (2.7)$$

whereas Λ is obtained as a solution of the equation (2.2). Discussion of this useful anzats (2.5) may be found in [5, 6].

2.1 Two-dimensional integrable systems with cubic integrals of motion

Let Q be a 2-dimensional Riemannian manifold, and its cotangent bundle T^*Q is naturally endowed with four canonical coordinates $q_{1,2}$ and $p_{1,2}$.

One of the natural Poisson bivectors P' (2.5) listed in [6] is defined by geodesic matrix depending on one variable q_2 :

$$\Pi = \begin{pmatrix} 0 & -\frac{i}{2} \left(\frac{\partial}{\partial q_2} + \frac{2h(q_2)}{g(q_2)} \right) \mathbb{F} \\ 0 & \mathbb{F} \end{pmatrix}, \quad \mathbb{F} = \left(g(q_2)p_2 - ih(q_2)p_1 \right)^2, \quad i = \sqrt{-1}, \quad (2.8)$$

and diagonal potential matrix

$$\Lambda = \alpha \exp \left(iq_1 - \int \frac{h(q_2)}{g(q_2)} dq_2 \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \alpha \in \mathbb{R}. \quad (2.9)$$

Here $g(q_2)$ and $h(q_2)$ are arbitrary functions and

$$x_{22} = -\frac{g(q_2)}{2h(q_2)}.$$

Other functions y_{ik} , x_{ik} and z_k in the definition (2.6) equal zero.

A characteristic polynomial of the corresponding recursion operator $N = P'P^{-1}$ reads as

$$\mathbb{B}(\eta) = \det(N - \eta I) = \left(\eta^2 - (\mathbb{F} + 2\Lambda_{11})\eta + \Lambda_{11}^2 \right)^2.$$

Here \mathbb{F} (2.8) is a complete square, it allows us to introduce the coordinates $v_{1,2}$ as roots of the linear in momenta $p_{1,2}$ polynomial

$$B(\lambda) = (\lambda - v_1)(\lambda - v_2) = \lambda^2 - i\sqrt{\mathbb{F}}\lambda + \Lambda_{11}. \quad (2.10)$$

In this coordinates $\mathbb{B} = (\eta + v_1^2)^2(\eta + v_2^2)^2$. We prefer to use the linear in momenta polynomial $B(\lambda)$ instead of $\mathbb{B}(\eta)$ because in this case it is easy to find a solution of the equations

$$\{B(\lambda), A(\mu)\} = \frac{\lambda}{\mu - \lambda} \left(\frac{B(\lambda)}{\lambda} - \frac{B(\mu)}{\mu} \right), \quad \{A(\lambda), A(\mu)\} = 0, \quad (2.11)$$

with respect to other linear in momenta polynomial

$$A(\lambda) = \int \frac{idq_2}{g(q_2)} - \frac{ip_1}{\lambda}.$$

Equations (2.11) entail that canonically conjugated momenta are equal to

$$p_{v_{1,2}} = A(\lambda = v_{1,2}). \quad (2.12)$$

Substituting variables

$$x = a_0 v_k^{-1}, \quad z = ip_k, \quad k = 1, 2, \quad a_0 \in \mathbb{R},$$

into the generic equation of the so-called (3,4) algebraic curve [1]

$$\Phi(z, x) = z^3 + (a_1x + a_2)z^2 + (H_1x^2 + b_1x + b_2)z + x^4 + H_2x^3 + c_1x^2 + c_2x + c_3 = 0, \quad a_k, b_k, c_k \in \mathbb{R}, \quad (2.13)$$

and solving the resulting equations with respect to $H_{1,2}$, one gets the following Hamilton function

$$H_1 = T + V + \frac{1}{a_0w_2} \left((a_1w_2^2 - b_1w_2 + c_2)h - 2a_1w_2 + b_1 \right) p_1 + \frac{ig(a_1w_2^2 - b_1w_2 + c_2)}{a_0w_2} p_2, \quad (2.14)$$

where geodesic Hamiltonian and potential are equal to

$$\begin{aligned} T &= \frac{1}{a_0^2w_2} \left((c_3 - w_2^3 + a_2w_2^2 - b_2w_2)h^2 + (b_2 + 3w_2^2 - 2a_2w_2)h - 3w_2 + a_2 \right) p_1^2 \\ &+ \frac{ig}{a_0^2w_2} \left(2(c_3 - w_2^3 + a_2w_2^2 - b_2w_2)h + b_2 + 3w_2^2 - 2a_2w_2 \right) p_1p_2 - \frac{g^2(c_3 - w_2^3 + a_2w_2^2 - b_2w_2)}{a_0^2w_2} p_2^2 \\ V &= -\frac{a_0^2e^{-iq_1}}{\alpha w_1w_2} - \frac{\alpha w_1(c_3 - w_2^3 + a_2w_2^2 - b_2w_2)e^{iq_1}}{a_0^2w_2} + \frac{c_1}{w_2}. \end{aligned}$$

Here

$$w_1 = \exp\left(-\int \frac{h(q_2)}{g(q_2)} dq_2\right), \quad w_2 = \int \frac{dq_2}{g(q_2)}.$$

Second integral of motion H_2 is a cubic polynomial in momenta $p_{1,2}$.

If $h \neq 0$ and $g \neq 0$, in order to reduce Hamiltonian H_1 (2.14) to the natural form, we have to solve two integral equations

$$a_1 w_2^2 - b_1 w_2 + c_2 = 0, \quad b_1 - 2a_1 w_2 = 0, \quad (2.15)$$

with respect to functions $h(q_2)$, $g(q_2)$ and parameters a_1, b_1, c_2 . If we want to obtain diagonal metric, then we have to add one more equation to this system. There is an additional freedom related with a possible canonical transformation $p_2 \rightarrow p_2 + f(q_2)$, which change linear in momenta terms in (2.14).

If $Q = \mathbb{S}^2$ is a two-dimensional unit sphere with spherical coordinates

$$q = (q_1, q_2) = (\phi, \theta) \quad \text{and} \quad p = (p_1, p_2) = (p_\phi, p_\theta), \quad (2.16)$$

then we could get a whole family of natural integrable systems on the sphere using different functions $h(\theta)$, $g(\theta)$ labeled by particular values of parameters in (2.15) [6, 9].

3 Natural Poisson bivectors in ζ -variables

In [7], solving the equation

$$\{H_1, H_2\} = 0$$

in framework of the Laplace method, author prefers to use special canonical coordinates

$$q_1 = \phi, \quad q_2 = \zeta, \quad p_1 = p_\phi, \quad p_2 = p_\zeta, \quad (3.17)$$

Here H_1 is a natural Hamilton function (1.1) with diagonal metric $g(\zeta)$ and H_2 is fixed by (1.2).

According to [7], special case $q = 0$ in (1.2) is rather difficult to obtain as the limit of the general case $q \neq 0$. So, we first integrate the special case at $q = 0$ and only then we consider the generic case. In contrast with [7], we restrict ourselves only by local analysis.

3.1 Case $q = 0$

Let us take integrals of motion from [7], Theorem 1 formulae (14-15):

$$H_1^{(0)} = \frac{1}{2} \left(F p_\zeta^2 + \frac{G}{4F} p_\phi^2 \right) + \lambda \sqrt{F} \cos \phi + \mu \zeta, \quad ({}' = D_\zeta) \quad (3.18)$$

$$H_2^{(0)} = p_\phi^3 - 2\lambda \left(\sqrt{F} \sin \phi p_\zeta + (\sqrt{F})' \cos \phi p_\phi \right) - 2\mu p_\phi,$$

with

$$F = -2\rho_0 + 3c_0\zeta + \zeta^3, \quad G = 9c_0^2 + 24\rho_0\zeta - 18c_0\zeta^2 - 3\zeta^4. \quad (3.19)$$

Here λ, μ, ρ_0 and c_0 are arbitrary parameters.

Substituting new variables (3.17) into definitions (2.8, 2.9) one gets a bivector P' depending on two arbitrary functions $h(\zeta)$ and $g(\zeta)$. Using this bivector we can easily solve the equation (2.3) and prove the following proposition.

Proposition 1 *Integrals of motion $H_{1,2}^{(0)}$ (3.18) are in bi-involution*

$$\{H_1^{(0)}, H_2^{(0)}\} = \{H_1^{(0)}, H_2^{(0)}\}' = 0 \quad (3.20)$$

with respect to a pair of compatible Poisson brackets associated with the canonical Poisson bivector P (2.1) and natural Poisson bivector P' (2.5) defined by

$$\Pi = \begin{pmatrix} 0 & \frac{i}{2} \left(\frac{\partial}{\partial \zeta} + \frac{F'}{F} \right) \mathbb{F} \\ 0 & \tilde{\mathbb{F}} \end{pmatrix}, \quad \mathbb{F} = - \left(2p_\zeta - \frac{iF'}{F} p_\phi \right)^2 \quad (3.21)$$

and

$$L = -\frac{4\lambda e^{-i\phi}}{\sqrt{F}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad x_{22} = -\frac{F}{F'}. \quad (3.22)$$

As a consequence, substituting $g(\zeta) = 2$ and $h(\zeta) = (\ln F)'$ into variables $v_{1,2}$ (2.10) and $p_{v_{1,2}}$ (2.12) one gets variables of separation for this integrable model.

Below, in order to extend the known palette of natural Poisson bivectors listed in [6], we consider another solution of the equation (2.2) depending on both variables ϕ and ζ .

Proposition 2 *Integrals of motion $H_{1,2}^{(0)}$ (3.18) are in the bi-involution (3.20) with respect to canonical Poisson bracket and bracket $\{.,\}'$ associated with the natural Poisson bivector \tilde{P}' (2.5) defined by 2×2 geodesic matrix*

$$\tilde{\Pi} = \begin{pmatrix} \tilde{\mathbb{F}} & -\frac{i\tilde{\mathbb{F}}'}{2} \\ 0 & 0 \end{pmatrix}, \quad \tilde{\mathbb{F}} = \left[e^{i\phi} \left(\frac{iF'}{2\sqrt{F}} p_\phi + \sqrt{F} p_\zeta \right) \right]^2, \quad (3.23)$$

diagonal potential matrix

$$\tilde{\Lambda} = \lambda e^{i\phi} \sqrt{F} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.24)$$

and function

$$y_{11} = -\frac{i}{2}.$$

Other functions y_{ik} , x_{ik} and z_k in the definition (2.6) equal zero.

As for the Lagrange top, this geodesic matrix (3.23) is factorized $\tilde{\Pi} = e^{2i\phi} \hat{\Pi}(\zeta)$, see discussion in [6].

We are now able to analyze the corresponding recursion operator $N = \tilde{P}' P^{-1}$. For instance, we can define the variables of separation $u_{1,2}$ as roots of the linear in momenta polynomial

$$\tilde{B}(\eta) = (\eta - u_1)(\eta - u_2) = \eta^2 - i\sqrt{\mathbb{F}} \eta + \Lambda_{11} \quad (3.25)$$

$$= \eta^2 - i e^{i\phi} \left(\frac{3i(c_0 + \zeta^2)}{2\sqrt{-2\rho_0 + 3c_0\zeta + \zeta^3}} p_\phi + \sqrt{-2\rho_0 + 3c_0\zeta + \zeta^3} p_\zeta \right) \eta + \lambda \sqrt{-2\rho_0 + 3c_0\zeta + \zeta^3} e^{i\phi},$$

so that the characteristic polynomial of N reads as $\det(N - \eta I) = (\eta - u_1^2)^2 (\eta - u_2^2)^2$. The conjugated momenta are equal to

$$p_{u_k} = \frac{i p_\phi}{u_k} - \frac{i \lambda \zeta}{u_k^2}, \quad k = 1, 2.$$

The inverse transformation looks like

$$\begin{aligned} \zeta &= -\frac{i u_1 u_2 (u_1 p_{u_1} - u_2 p_{u_2})}{\lambda (u_1 - u_2)}, & p_\phi &= \frac{i(u_1^2 p_{u_1} - u_2^2 p_{u_2})}{u_1 - u_2}, \\ \phi &= \frac{i}{2} \ln \left(\frac{\lambda^2 F}{u_1^2 u_2^2} \right), & p_\zeta &= -\frac{i F'}{2F} p_\phi - \frac{i \lambda (u_1 + u_2)}{u_1 u_2}. \end{aligned} \quad (3.26)$$

In these variables of separation our initial bivector P' (3.21,3.22) looks like

$$P' = 3 \begin{pmatrix} 0 & 0 & u_1^2 p_{u_1}^2 & 0 \\ 0 & 0 & 0 & u_2^2 p_{u_2}^2 \\ -u_1^2 p_{u_1}^2 & 0 & 0 & 0 \\ 0 & -u_2^2 p_{u_2}^2 & 0 & 0 \end{pmatrix}.$$

Now we can calculate the integrals of motion

$$\begin{aligned} H_1^{(0)} &= -\frac{i\mu u_1 u_2 (u_1 p_{u_1} - u_2 p_{u_2})}{\lambda(u_1 - u_2)} + \frac{(u_2^2 + u_1 u_2 + u_1^2)\rho_0 \lambda^2}{u_1^2 u_2^2} - \frac{3ic_0(u_1 p_{u_2} - p_{u_1} u_2)\lambda}{2(u_1 - u_2)} + \frac{u_1 u_2}{2} \\ &\quad - \frac{i(u_1^2 p_{u_1}^2 + u_2 p_{u_2} u_1 p_{u_1} + u_2^2 p_{u_2}^2)(u_1 p_{u_1} - u_2 p_{u_2})u_1 u_2}{2(u_1 - u_2)\lambda}, \\ H_2^{(0)} &= -\frac{2i\mu(p_{u_1} u_1^2 - p_{u_2} u_2^2)}{u_1 - u_2} + \frac{2(u_1 + u_2)\rho_0 \lambda^3}{u_1^2 u_2^2} + \frac{3ic_0(p_{u_1} - p_{u_2})\lambda^2}{u_1 - u_2} + \frac{u_1 + u_2}{\lambda} - \frac{i(p_{u_1}^3 u_1^4 - u_2^4 p_{u_2}^3)}{u_1 - u_2}. \end{aligned}$$

and the control matrix

$$\mathbf{F} = \begin{pmatrix} 0 & \frac{u_1 u_2}{2\lambda} \\ -2\lambda & u_1 + u_2 \end{pmatrix}, \quad \text{from the definition} \quad P' dH_i^{(0)} = P \sum_{j=1}^2 \mathbf{F}_{ij} dH_j^{(0)}, \quad i = 1, \dots, 2.$$

Suitable normalized left eigenvectors of control matrix \mathbf{F} form the Stäckel matrix \mathbf{S} , so the notion of \mathbf{F} allows us to compute the corresponding separated relations

$$\sum_{j=1}^2 \mathbf{S}_{ij} H_j + U_i = 0, \quad i = 1, 2.$$

Here functions \mathbf{S}_{ij} and U_i depend only on one pair (u_i, p_{u_i}) of canonical variables of separation [3].

In our case the integrals of motion and the variables of separation are related via the following separated relations

$$\Phi(u, z) = z^3 + (3\lambda^2 c_0 - 2\mu u^2)z + \lambda u^4 - H_2^{(0)} u^3 + 2\lambda H_1^{(0)} u^2 - 2\lambda^3 \rho_0 = 0, \quad (3.27)$$

at $u = u_{1,2}$ and $z = iu_{1,2}^2 p_{1,2}$. Equation $\Phi(u, z) = 0$ defines genus three non-hyperelliptic curve with the following base of the holomorphic differentials

$$\Omega_1 = \frac{du}{2\mu u^2 - 3\lambda^2 c_0 - 3z^2}, \quad \Omega_2 = \frac{u du}{2\mu u^2 - 3\lambda^2 c_0 - 3z^2}, \quad \Omega_3 = \frac{z du}{2\mu u^2 - 3\lambda^2 c_0 - 3z^2}.$$

According to [1] it is the so-called (3,4) trigonal curve. It is easy to prove that the equations of motion in variables of separation have the following form

$$\begin{aligned} \frac{\dot{u}_1}{2\mu u_1^2 - 3\lambda^2 c_0 - 3z_1^2} + \frac{\dot{u}_2}{2\mu u_2^2 - 3\lambda^2 c_0 - 3z_2^2} &= \frac{i}{2\lambda}, \quad z_k = iu_k^2 p_k, \\ \frac{u_1 \dot{u}_1}{2\mu u_1^2 - 3\lambda^2 c_0 - 3z_1^2} + \frac{u_2 \dot{u}_2}{2\mu u_2^2 - 3\lambda^2 c_0 - 3z_2^2} &= 0. \end{aligned}$$

The aforementioned quadratures in the integral form

$$\int_{u_0}^{u_1} \Omega_1 + \int_{u_0}^{u_2} \Omega_1 = \frac{i}{2\lambda} t + \beta_1, \quad \int_{u_0}^{u_1} \Omega_2 + \int_{u_0}^{u_2} \Omega_2 = \beta_2, \quad (3.28)$$

represent the Abel-Jacobi map associated to the genus three non-hyperelliptic curve defined by the equation $\Phi(q, p) = 0$ (3.27). In particular it means that instead of z in $\Omega_{1,2}$ (3.28) we have to substitute the function on u obtained from the separated relation (3.27).

3.2 Case $q \neq 0$

Let us take integrals of motion from [7], Theorem 5 formulae (39-40):

$$H_1 = \frac{1}{2\zeta} \left(F p_\zeta^2 + \frac{G}{4F} p_\phi^2 \right) + \frac{\sqrt{F}}{2q\zeta} \cos \phi + \frac{\beta_0}{2q\zeta}, \quad (3.29)$$

$$H_2 = p p_\phi^3 + 2q H_1 p_\phi - \sqrt{F} \sin \phi p_\zeta - (\sqrt{F})' \cos \phi p_\phi,$$

with

$$F = c_0 + c_1 \zeta + c_2 \zeta^2 + c_3 \zeta^3, \quad G = F'^2 - 2FF'', \quad c_3 = \frac{p}{q}, \quad (3.30)$$

where β_0, c_0, c_1, c_2 and p, q are arbitrary parameters.

Proposition 3 *After substitution of the new function F (3.30) into the previous definitions (3.21, 3.22) and (3.23, 3.24) one gets two compatible Poisson bivectors P' and \tilde{P}' , so that the integrals of motion $H_{1,2}$ (3.29) are in the involution with respect to the corresponding Poisson brackets.*

Other calculations are standard. Namely, the variables of separation are given by

$$\begin{aligned} \tilde{B} &= (\eta - u_1)(\eta - u_2) = \eta^2 - e^{i\phi} \left(\frac{iF'}{2\sqrt{F}} p_\phi + \sqrt{F} p_\zeta \right) \eta - \frac{\sqrt{F} e^{i\phi}}{2q}, \\ p_{u_k} &= -\frac{i p_\phi}{u_k} - \frac{\zeta}{2q u_k^2}, \end{aligned} \quad (3.31)$$

whereas the inverse transformation reads as

$$\begin{aligned} \zeta &= \frac{2qu_1 u_2 (u_1 p_{u_1} - u_2 p_{u_2})}{u_1 - u_2}, & p_\phi &= \frac{i(u_1^2 p_{u_1} - u_2^2 p_{u_2})}{u_1 - u_2}, \\ \phi &= -\frac{i}{2} \ln \left(\frac{4q^2 u_1^2 u_2^2}{F} \right), & p_\zeta &= -\frac{iF'}{2F} p_\phi - \frac{u_1 + u_2}{2qu_1 u_2}. \end{aligned} \quad (3.32)$$

Then we have to calculate the control matrix \mathbf{F} and the corresponding Stäckel matrix \mathbf{S} in order to get the desired separated relations, which are obtained by substituting $u = u_{1,2}$ and $z = p u_{1,2}^2 p_{u_{1,2}}$ in the following equation

$$\Phi = 2z^3 - c_2 z^2 - \frac{(8q^2 H_1 u^2 - c_1)p}{2q} z - p^2 u^4 - 2ip^2 H_2 u^3 - \frac{\beta_0 p^2 u^2}{q} - \frac{p^2 c_0}{4q^2} = 0. \quad (3.33)$$

This equation $\Phi(u, z) = 0$ defines the genus three non-hyperelliptic curve (so-called (3,4)-curve, see [1]) with the holomorphic differentials

$$\Omega_1 = \frac{du}{-2c_3 \partial \Phi / \partial z}, \quad \Omega_2 = \frac{u du}{-2c_3 \partial \Phi / \partial z}, \quad \Omega_3 = \frac{z du}{-2c_3 \partial \Phi / \partial z}. \quad (3.34)$$

The desired quadratures read as

$$\frac{u \dot{u}_1}{8p^2 H_1 u_1^2 - 12c_3 z_1^2 + 4c_2 c_3 z_1 - c_1 c_3^2} + \frac{u \dot{u}_2}{8p^2 H_1 u_2^2 - 12c_3 z_2^2 + 4c_2 c_3 z_2 - c_1 c_3^2} = 0, \quad z_k = p u_k^2 p_k,$$

and

$$\frac{z_1 \dot{u}_1}{8p^2 H_1 u_1^2 - 12c_3 z_1^2 + 4c_2 c_3 z_1 - c_1 c_3^2} + \frac{z_2 \dot{u}_2}{8p^2 H_1 u_2^2 - 12c_3 z_2^2 + 4c_2 c_3 z_2 - c_1 c_3^2} = -\frac{p}{8}.$$

The above quadratures in the integral form

$$\int_{u_0}^{u_1} \Omega_2 + \int_{u_0}^{u_2} \Omega_2 = \beta_1, \quad \int_{u_0}^{u_1} \Omega_3 + \int_{u_0}^{u_2} \Omega_3 = -\frac{p}{8} t + \beta_2, \quad (3.35)$$

represent the Abel-Jacobi map associated to the genus three non-hyperelliptic curve defined by (3.33). If we change $u \rightarrow iu$, that corresponds to transformation $P' \rightarrow -P'$, one gets the equation $\Phi(u, z)$ (3.33) with real coefficients, but the coefficient before time variable in (3.28) becomes imaginary number.

In order to give an explicit theta-functions solution, one could apply the standard Weierstrass machinery describing the inversion of the hyperelliptic quadratures [10]. However, for the genus three trigonal curve such solution of the Jacobi inversion problem is only created. Some of the results referring to Weierstrass theory for the general trigonal curve of genus three may be found in [1].

On the other hand, for a non-hyperelliptic Riemann surface of genus three the moduli spaces of rank two bundles compactifies into a singular projective variety, which is closely related to the Kummer variety and which is a quartic hypersurface of \mathbb{P}^7 . We can try to apply the generic theory of vector bundles on algebraic varieties to investigation of this integrable system and vice versa [8].

3.3 Change of the time

If the dimension of the configurational space n is less than genus g of the separated curve, then the Abel map is either lack of uniqueness or degenerate. In this case we are free to choose n integrals of motion from the coefficients of the separated curve, that equivalent to choose the corresponding time variables. The Kepler change of the time, the Liouville transformations, the Maupertuis–Jacobi transformations and the coupling constant metamorphosis are examples of such duality. One of the well-known examples is a duality between harmonic oscillator and the Kepler system, which was expanded on the quantum case by Schrodinger.

All the known such transformations are associated only with the hyperelliptic curves, see [3, 4]. Here we want to show one example related with non-hyperelliptic curve. Let us consider the following equation

$$\Phi = 2z^3 - c_2 z^2 - \frac{(8q^2 \alpha u^2 - c_1)p}{2q} z - p^2 u^4 - 2ip^2 \hat{H}_2 u^3 + 2p^2 \hat{H}_1 u^2 - \frac{p^2 c_0}{4q^2} = 0, \quad (3.36)$$

which is related with the initial equation (3.33) by permutation of coefficients α and \hat{H}_1 only. If we substitute variables (3.31) into (3.36) and solve the resulting pair of separated equations, then one gets the Hamilton function

$$\begin{aligned} \hat{H}_1 &= \zeta(H_1 + \alpha) + \frac{\beta_0}{2q} = \frac{1}{2} \left(F p_\zeta^2 + \frac{G}{4F} p_\phi^2 \right) + \frac{c_3 \sqrt{F}}{2p} \cos \phi - \alpha \zeta, \\ F &= c_0 + c_1 \zeta + c_2 \zeta^2 + c_3 \zeta^3, \quad G = F'^2 - 2FF'. \end{aligned} \quad (3.37)$$

Second integral of motion in this case is equal to

$$\hat{H}_2 = p p_\phi^3 - \frac{F'}{2\sqrt{F}} p_\phi \cos \phi - \sqrt{F} p_\zeta \sin \phi + \frac{2\alpha p}{c_3} p_\phi.$$

Integrals $\hat{H}_{1,2}$ may be obtained from $H_{1,2}^{(0)}$ (3.18) at special values of parameters using canonical transformation $\zeta \rightarrow \zeta - c_2/3c_3$.

Dual Stäckel systems with Hamiltonians H_1 and \hat{H}_1 are related to each other by a canonical transformation of the extended phase space [3, 4]. For the both systems trigonal curve defined by (3.33) or (3.36) has the same holomorphic differentials Ω_k (3.34), but instead of quadratures (3.35) in the second case we have

$$\int_{u_0}^{u_1} \Omega_1 + \int_{u_0}^{u_2} \Omega_1 = \frac{1}{4pc_3} t + \beta_1, \quad \int_{u_0}^{u_1} \Omega_2 + \int_{u_0}^{u_2} \Omega_2 = \beta_2, \quad (3.38)$$

Similar to hyperelliptic case, this duality (change of the time) is related with the different choice of the holomorphic differentials in the inversion Jacobi problem [3, 4].

Using the same permutation of coefficients in the generic equation (2.13) of (3,4)-curve we can obtain the dual Hamiltonian to more complicated Hamilton function (2.14). Other possible generalization is related with another pair of differentials in the Abel map:

1. case $q = 0$ is associated with Ω_1 and Ω_2 (3.38);

2. case $q \neq 0$ is associated with Ω_2 and Ω_3 (3.35);
3. one more integrable case is associated with Ω_1 and Ω_3 .

Some difficulty here is related with the coordinates choice, because we can not directly recognize interesting physical models in ζ -variables. So, different possible generalizations of the Goryachev, Chaplygin, Dullin-Matveev systems and other systems from [6, 9] associated with the generic (3,4)-curve in term of physical variables on the two-dimensional sphere will be discussed in a forthcoming publication.

According to [6], there are natural generalizations of matrices Π and Λ (2.8,2.9) on the 3-dimensional case. It can allow as to obtain a new three-dimensional natural integrable system with higher order integrals of motion constructively.

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